

THE TOPOLOGICAL COMPLEXITY OF CANTOR ATTRACTORS IN ONE-DIMENSIONAL DYNAMICS

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ABSTRACT. For a C^3 unimodal map with a non-flat critical point, the topological complexity of Cantor attractor is considered. If the map has a Cantor attractor, then for any open cover \mathcal{U} of $\omega(c)$, there exists a constant $C > 0$ such that the complexity function $p(\mathcal{U}, n)$ is less than $Cn \log n$.

1. INTRODUCTION

Topological entropy is a useful invariant for measuring the complexity of a dynamical system. It describes the exponential growth rate of the orbits of the systems. For a system with zero entropy, several notions such as complexity function, sequence entropy, entropy dimension were introduced to measure the sub-exponential growth rate, see for example [1, 9, 8].

Let X be a compact metric space and $f : X \rightarrow X$ be a continuous map. For an open cover \mathcal{U} of X , let $N(\mathcal{U})$ be the minimal cardinality of a sub-cover of \mathcal{U} . For open covers \mathcal{U}, \mathcal{V} of X , let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. The *topological complexity function* of an open cover \mathcal{U} is the non-decreasing function

$$p(\mathcal{U}, n) = N\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}\right), \quad n = 1, 2, \dots$$

The complexity function can be used to characterize the dynamical behavior of some systems. For example, it is well known [1] that a system is equicontinuous if and only if the complexity function is bounded for each open cover.

In this paper, we will consider the topological complexity of a smooth interval map restricted to an invariant Cantor set. A C^1 map $f : [0, 1] \rightarrow [0, 1]$ is called *unimodal* if there exists a unique $c \in (0, 1)$ (called the *critical point*) such that $f'(c) = 0$ and such that f' has different signs on the components of $[0, 1] \setminus \{c\}$. We shall always assume that f is C^3 outside c and c is non-flat, i.e., there exist C^3 local diffeomorphisms ϕ, ψ defined on a neighborhood of 0 with $\phi(0) = c$, $\psi(0) = f(c)$, and a real number $\ell = \ell_c > 1$ (called the *order* of c), such that $|\psi^{-1} \circ f \circ \phi(x)| = |x|^\ell$ holds when $|x|$ is small. Let \mathcal{U} denote the collection of unimodal maps with the above properties and let \mathcal{U}_* denote the collection of $f \in \mathcal{U}$ which have all periodic points hyperbolic repelling.

The notion of attractor was introduced in [16]. Let f be a unimodal map. A (minimal) *metric attractor* is a compact invariant subset $A \subset [0, 1]$ such that $\{x \in [0, 1] : \omega(x) \subset A\}$ has positive Lebesgue measure, but no invariant compact proper subset of A has this property. It is known that a metric attractor of $f \in \mathcal{U}$ can be one of the following form: an attracting periodic orbit, or the union of a cycle of periodic intervals, or a Cantor set. In the last case, the Cantor attractor must coincide with $\omega(c)$. See [15].

Main Theorem. *Let $f : [0, 1] \rightarrow [0, 1]$ be a C^3 unimodal map with a non-flat critical point c . Suppose that $\omega(c)$ is a Cantor attractor. Then for each open cover \mathcal{U} of $\omega(c)$, there is a constant $C > 0$ such that the complexity function of $f|_{\omega(c)}$ satisfies $p(\mathcal{U}, n) \leq Cn \log n$ for $n > 1$.*

This theorem strengthens a result of Blokh and Lyubich [3, Section 11] which asserts that $f|_{\omega(c)}$ has topological entropy zero. In fact, our main theorem clearly implies that the topological entropy dimension of $f|_{\omega(c)}$ is zero (see [8] for the notion of entropy dimension). Note also a different “low complexity” property was proved for Cantor attractors in [6]. That result is independent of ours and has an interesting consequence that $f : \omega(c) \rightarrow \omega(c)$ is uniquely ergodic.

Recall that a unimodal map f is called *renormalizable* if there exists an interval I which contains the critical point c in its interior, and a positive integer $s > 1$, such that the intervals $I, f(I), \dots, f^{s-1}(I)$ have pairwise disjoint interiors, $f^s(I) \subset I$, and $f^s(\partial I) \subset \partial I$. The unimodal map $f^s : I \rightarrow I$ is called a *renormalization* of period s . If there are restrictive intervals with arbitrarily large period, then f is called *infinitely renormalizable*.

When $f \in \mathcal{U}_*$ is infinitely renormalizable, our theorem is well-known. In fact, a much stronger statement is true. In this case, $\omega(c)$ is a minimal Cantor set and $f|_{\omega(c)}$ is equicontinuous, and by the theorem in [1] the complexity function is bounded for each open cover. This fact can also be obtained more directly: if J is a restricted interval with period s , then $\mathcal{U} = \{int(J), f(int(J)), \dots, f^{s-1}(int(J))\}$ is an open cover of $\omega(c)$, and $p(\mathcal{U}, n) = s$. Since the length of restricted intervals goes to zero, it follows that the complexity function is bounded for each open cover.

In the case that f is not infinitely renormalizable, the Cantor attractor $\omega(c)$ is usually called a *wild attractor* because its basin of attraction is a meager set. The theorem follows from an upper bound for the number of children of nice intervals. See the Reduced Main Theorem.

A system is called a null system if the sequence entropy is zero for every sequence. It is proved in [9] that a minimal system is null if and only if it is an almost one to one extension of an equicontinuous system. It seems that all known examples of wild attractor are null systems, see [4, 12] for Fibonacci maps and [2] for tent maps. It is an interesting problem whether every wild attractor is null.

2. NICE INTERVALS AND CHILDREN

Consider a map $f : [0, 1] \rightarrow [0, 1]$ be a unimodal map in \mathcal{U}_* . Let c denote the critical point of f and let ℓ be the order of c . Without loss of generality, we may assume $f(0) = f(1) = 0$. We will also assume that f is geometrically symmetric near c .

2.1. Notations and terminologies. Given a subset V of $[0, 1]$ and an integer $k \geq 0$, we say that a component J of $f^{-k}(V)$ is a *pull back of V by f^k* . We say that such a pull back is

- *critical* if it contains the critical point c ;
- *diffeomorphic* if f^k maps J diffeomorphically onto a component of V ;
- *unimodal* if $J \ni c$ and f^{k-1} maps a neighborhood of $f(J)$ diffeomorphically onto a component of V .

For $T \subset [0, 1]$, let

$$D(T) = \{x \in [0, 1] : f^k(x) \in T \text{ for some } k \geq 1\}.$$

The *first entry map* $R_T : D(T) \rightarrow T$ is defined as $x \mapsto f^{k(x)}(x)$, where $k(x)$ is the *entry time* of x into T , i.e., the minimal positive integer such that $f^{k(x)}(x) \in T$. The map $R_T|_{(D(T) \cap T)}$ is called the *first return map* of T . A component of $D(T)$ (resp. $D(T) \cap T$) is called an *entry domain* (resp. *return domain*) of T . Let $\mathcal{L}_x(T)$ denote the entry domain containing x .

Let us call an open set $T \subset [0, 1]$ *nice* if $f^n(\partial T) \cap T = \emptyset$ for all $n \geq 0$ and T does not contain a fixed point of f . It is well-known that for such an open set T ,

- pull-backs of a nice set are again nice;
- if J_j is a pull back of T by f^{k_j} , $j = 1, 2$, and $k_1 \geq k_2$, then $J_1 \cap J_2 = \emptyset$ or $J_1 \subset J_2$;
- the entry time is constant in any component of $D(T)$, so the first entry map $R_T : D(T) \rightarrow T$ is continuous.

Moreover, if $f \in \mathcal{U}_*$, then there exists an arbitrarily small symmetric nice interval $T \ni c$. See for example [14].

A nice interval $T \ni c$ is called *symmetric* if $f(\partial T)$ consists of a single point. A unimodal pull back of a nice interval $T \ni c$ is also called a *child* of T .

We say that f is *persistently recurrent* if for each symmetric nice interval $T \ni c$, the number of children of T is finite. The following is well-known.

Proposition 1 (Blok-Lyubich [3]). *Suppose that $f \in \mathcal{U}_*$ has a Cantor attractor A . Then $A = \omega(c) \ni c$, A is a minimal set and f is persistently recurrent.*

2.2. The Reduced Main Theorem.

Reduced Main Theorem. *Suppose that $f \in \mathcal{U}_*$ is non-renormalizable and that $\omega(c)$ is a Cantor attractor. For each symmetric nice interval $Y \ni c$, there exists $n_0 = n_0(Y) \geq 2$ such that if T is a critical pull back of Y by f^n for some $n \geq n_0$, then the number of children of T is bounded from above by $C \log n$, where $C > 0$ is a constant depending only the critical order.*

Proof of the Main Theorem. We may assume that f is non-renormalizable, as in the infinitely renormalizable case the Main Theorem is easy to see, and the finitely renormalizable case can be reduced to the non-renormalizable case.

Given a nice interval $Y \ni c$, let \mathcal{Y} denote the collection of all components of $D(Y) \cup Y$ which intersect $\omega(c)$. Since $f : \omega(c) \rightarrow \omega(c)$ is minimal, \mathcal{Y} is a finite cover of $\omega(c)$. Since f has no wandering interval, see [15], the maximal length of elements of \mathcal{Y} tends to zero as $|Y| \rightarrow 0$. Thus for any open cover \mathcal{U} of $\omega(c)$, there exists a small symmetric nice interval $Y \ni c$ such that \mathcal{Y} is a refinement of \mathcal{U} , hence

$$(1) \quad p(\mathcal{U}, n) \leq p(\mathcal{Y}, n) = N\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{Y}\right) = N(f^{-(n-1)}\mathcal{Y}).$$

Let $n_0 = n_0(Y)$ be given by the Reduced Main Theorem. Let us show that for each $n \geq n_0$, the number of components of $f^{-n}(Y)$ intersecting $\omega(c)$ is bounded from above by $C_1 n \log n$, where $C_1 > 0$ is a constant. Indeed, for each component J of $f^{-n}(Y)$, there exists a minimal integer $n' = n'(J) \in \{0, 1, \dots, n\}$ such that $f^{n'}(J)$ contains the critical point c . Let $\mathcal{J}_{n'}$ denote the collection of all components J of $f^{-n}(Y)$ with $n'(J) = n'$ and $J \cap \omega(c) \neq \emptyset$. Clearly, \mathcal{J}_0 has at most one element. Consider $n' > 0$ with $\mathcal{J}_{n'} \neq \emptyset$. Then $f^{n-n'}(c) \in Y$. Let T be the component of $f^{n'-n}(Y)$ which contains c . By the Reduced Main Theorem, if $n - n' \geq n_0$, then T has at most $C \log n$ children. If $n - n' < n_0$, then by Proposition 1, there exists $M = M(Y) > 0$ such that the number of children of T is at most M . For each $J \in \mathcal{J}_{n'}$, $f^{n'}$ maps J diffeomorphically onto T . So if $t = t(J)$ is the first entry time of c to J , then the component of $f^{-t}(J)$ which contains c is a child of T . Moreover, different J 's correspond to different children. Thus $\#\mathcal{J}_{n'} \leq \max(C \log n, M)$. Therefore, the total number of $f^{-n}(Y)$ intersecting $\omega(c)$ is bounded by $C_1 n \log n$ for some $C_1 > 0$.

Let $m_0 \geq 1$ be such that each element of \mathcal{Y} is a pull back of Y by f^m for some $0 \leq m < m_0$. Thus for $n \geq 2$,

$$N(f^{-(n-1)}\mathcal{Y}) \leq C_1(n + m_0 - 1) \log(n + m_0 - 1) \leq C_2 n \log n,$$

which, together with (1), implies that $p(\mathcal{U}, n)$ is of order $n \log n$. \square

Remark 1. In [7], it is proved that a Fibonacci-like unimodal map has sub-linear complexity, i.e., $p(\mathcal{U}, n) \leq Cn$ for some constant $C > 0$ and each open cover \mathcal{U} . For a Fibonacci-like unimodal map, the children of nice intervals are bounded by a constant. Therefore their result is compatible with ours.

It is not clear to us whether the upper bounds appearing above are optimal. Indeed, the following simpler problem is open:

Problem. *Give a positive integer $N \geq 2$. Does there exist a real number ℓ_0 such that if $f \in \mathcal{U}_*$ has critical order $\ell > \ell_0$ and satisfies the following property: each nice interval has at most N children, then f has a wild attractor?*

The rest of this paper is devoted to the proof of the Reduced Main Theorem. In § 3, we study the size of children of a given nice interval and the geometry of their return domains. In §4, we study the distortion of “empty space” under pull backs. It should be noted that the presence of central cascade is responsible for most complications of the arguments in both sections. The proof of the Reduced Main Theorem is completed in §5.

3. REAL BOUNDS

Consider a map $f \in \mathcal{U}_*$ with a recurrent critical point c . We say a constant is *universal* if it depends only on ℓ . In this section, we shall obtain upper bounds of length of children of given nice intervals and the geometry of their return domains. The main result is Proposition 2.

3.1. Preliminaries. Given a bounded interval I and a constant $\tau > 0$, let τI denote the open interval which is concentric with I and has length $\tau|I|$. We say that a bounded interval J is τ -well inside an interval I if $I \supset (1 + 2\tau)J$, i.e., both components of $I \setminus J$ have length at least $\tau|J|$.

The Koebe principle is the main tool to control distortion in one-dimensional dynamics. The following version was taken from [5, Proposition 1], whose proof is based on previous results in the literature, in particular [17].

Theorem 1. *There exists $\eta(f) > 0$ such that the following holds. Let $s \geq 1$ be an integer and let T be an interval. Assume that $f^s|_T$ is a diffeomorphism onto its image and that $|f^s(T)| < \eta(f)$. If J is a subinterval of T such that $f^s(J)$ is τ -well inside $f^s(T)$, then*

1. *for any $x, y \in J$,*

$$0.9 \left(\frac{\tau}{1 + \tau} \right)^2 \leq \frac{|Df^s(x)|}{|Df^s(y)|} \leq \frac{1}{0.9} \left(\frac{1 + \tau}{\tau} \right)^2;$$

2. *J is τ' -well inside T , where $\tau' = \frac{0.9\tau^2}{1 + 2\tau}$.*

Given a symmetric nice interval $I \ni c$, we shall use the following notation: $I^0 = I$ and I^{k+1} is the return domain of I^k that contains c . The sequence

$$I^0 \supset I^1 \supset I^2 \supset \dots,$$

is often called the *principal nest* starting from I . The first return map $R_{I^n} : I^{n+1} \rightarrow I^n$ is called *central* if $R_{I^n}(c) \in I^{n+1}$ and *non-central* otherwise. We say that $R_{I^n} : I^{n+1} \rightarrow I^n$ is *high* if $R_{I^n}(I^{n+1}) \ni c$ and *low* otherwise.

The following Real Bounds theorem was first proved by Martens [14] in the case that f has negative Schwarzian derivative, and extended to general smooth unimodal maps in [10].

Theorem 2. *There exists a universal constants $\rho > 0$ such that for any small symmetric nice interval $I^0 \ni c$, the following hold:*

- (i) *If $R_{I^0} : I^1 \rightarrow I^0$ is non-central and low, then I^1 is ρ -well inside I^0 ;*
- (ii) *If $R_{I^0} : I^1 \rightarrow I^0$ is non-central and high, then I^2 is ρ -well inside I^1 ;*
- (iii) *If I^1 is not ρ -well inside I^0 , then f^{s-1} maps a neighborhood of $f(I^1)$ diffeomorphically onto a ρ -scaled neighborhood of I^0 , where s is the return time of c into I^0 . In particular, the map $f^{s-1}|_{f(I^1)}$ has uniformly bounded distortion: for any $x, y \in I^1$,*

$$|Df^{s-1}(f(x))| \leq K(\rho)|Df^{s-1}(f(y))|,$$

where $K(\rho) > 1$ is a constant.

A sequence of open intervals $\{T_j\}_{j=0}^s$ is called a *chain* if for each $j = 0, 1, \dots, s-1$, T_j is a component of $f^{-1}(T_{j+1})$. The *order* of the chain is the number of j 's with $0 \leq j < s$ such that T_j contains the critical point c .

The following theorem is an improvement of Theorem C(1) in [17] for unimodal maps, which gives relationship between the constants τ and τ' .

Theorem 3. *Assume that f is not infinitely renormalizable. For any $\tau > 0$ there exists $\tau' > 0$, such that the following holds. Let $c \in J \subset I$ be small symmetric nice intervals such that J is τ -well inside I . Then for any $x \in D(J)$, $\mathcal{L}_x(J)$ is τ' -well inside $\mathcal{L}_x(I)$. Moreover, for each constant $\tau_* > 0$ there exist constants $C = C(\tau_*) > 0, \alpha = \alpha(\tau_*) > 0$ such that if $\tau > \tau_*$, then we can choose τ' such that*

$$(2) \quad \tau' \geq C\tau^\alpha.$$

Proof. By Theorem 1 and non-flatness of the critical point, it suffices to prove the statement for $x \in D(J) \setminus I$. Let $I^0 = I$. Let $m(0) = 0$ and $1 \leq m(1) < m(2) < \dots$ be all the non-central return moments, i.e., the return map $R_{I^{m(k)-1}}$ is non-central. Since f is not infinitely renormalizable, $|I^n| \rightarrow 0$ as $n \rightarrow \infty$, provided that I is small enough. So there exists $k \geq 0$ such that

$$I^0 \supset I^{m(1)} \supset \dots \supset I^{m(k)} \supsetneq J \subset I^{m(k+1)}.$$

Define τ_i , $1 \leq i \leq k+1$ such that

$$\frac{|I^{m(i-1)}|}{|I^{m(i)}|} := 1 + 2\tau_i, \text{ for } 1 \leq i \leq k,$$

and

$$\frac{|I^{m(k)}|}{|J|} := 1 + 2\tau_{k+1}.$$

Then

$$(3) \quad 1 + 2\tau = \frac{|I|}{|J|} = \prod_{i=1}^k \frac{|I^{m(i-1)}|}{|I^{m(i)}|} \cdot \frac{|I^{m(k)}|}{|J|} = \prod_{i=1}^{k+1} (1 + 2\tau_i).$$

For each $1 \leq i \leq k$, the first entry map $R_{I^{m(i)}} : \mathcal{L}_x(I^{m(i)}) \rightarrow I^{m(i)}$ can be extended diffeomorphically onto $I^{m(i-1)}$ (see Lemma 4). By Theorem 1, $\mathcal{L}_x(I^{m(i)})$ is τ'_i -well inside $\mathcal{L}_x(I^{m(k-1)})$, where $\tau'_i = 0.9 \frac{\tau_i^2}{1+2\tau_i}$. Similarly, since $I^{m(k)} \supset J \supset I^{m(k+1)}$, the first entry map $R_J : \mathcal{L}_x(J) \rightarrow J$ can be extended diffeomorphically onto $I^{m(k)}$, and $\mathcal{L}_x(J)$ is τ'_{k+1} -well inside $\mathcal{L}_x(I^{m(k)})$, where $\tau'_{k+1} = 0.9 \frac{\tau_{k+1}^2}{1+2\tau_{k+1}}$. In conclusion, $\mathcal{L}_x(J)$ is τ' -well inside $\mathcal{L}_x(I)$, where

$$(4) \quad 1 + 2\tau' = \prod_{i=1}^{k+1} (1 + 2\tau'_i).$$

Let us prove that τ' is bounded away from zero. By Theorem 2, for each $2 \leq i \leq k$, we have $\tau_i \geq \rho$. So we are done if $k \geq 2$. If $k \leq 1$, then by (3), $(1 + 2\tau_i)^2 \geq 1 + 2\tau$ holds for $i = 1$ or 2 , thus τ' is bounded from below by a positive constant depending on τ .

Now assume τ is bounded from below by a constant $\tau_* > 0$ and let us prove (2). Let $\rho_* = \min(\tau_*, \rho)$, and let

$$\mathcal{I} = \{1 \leq i \leq k+1 : (1 + 2\tau_i)^4 > 1 + 2\rho_*\}.$$

Then $\{2, \dots, k\} \subset \mathcal{I}$. So by (3),

$$(5) \quad \prod_{i \in \mathcal{I}} (1 + 2\tau_i) \geq \frac{1 + 2\tau}{\sqrt{1 + 2\rho_*}} \geq \sqrt{1 + 2\tau}.$$

For each $i \in \mathcal{I}$, τ'_i is bounded away from zero, so there exists a constant $\mu \in (0, 1)$ such that $\tau'_i \geq \mu\tau_i$. Thus by (4),

$$1 + 2\tau' \geq \prod_{i \in \mathcal{I}} (1 + 2\mu\tau_i) \geq \prod_{i \in \mathcal{I}} (1 + 2\tau_i)^\mu.$$

Together with (5), this implies

$$1 + 2\tau' \geq (1 + 2\tau)^{\mu/2}.$$

The inequality (2) follows. \square

Recall that a child $J \ni c$ of a symmetric nice $I \ni c$ is a unimodal pull back of I by f^s for some $s \geq 1$. The integer s is called a *transition time* from J to I .

Lemma 1. *Let J be a child of I with transition time s , then for each $x \in J$, the return time of x to J is not less than s .*

Proof. Let $\{J_i\}_{i=0}^s$ be the chain with $J_0 = J$ and $J_s = I$. Since $f^{s-1} : J_1 \rightarrow J_s$ is diffeomorphic, $c \notin J_i$ for each $1 \leq i \leq s-1$. Therefore $J_i \cap J = \emptyset$ ($1 \leq i \leq s-1$), since otherwise $J_i \supset J \ni c$, which is impossible. For each $x \in J$, $f^i(x) \in J_i$ ($1 \leq i \leq s-1$). If $f^k(x) \in J$, then $k \geq s$. \square

Lemma 2. *Let $I \ni c$ be a small nice interval and let J be a child of I . Assume that J is τ -well inside I . Then J is a τ' -nice interval, where $\tau' > 0$ depends only on τ . Moreover, when τ is large enough, then $\tau' \geq C\tau^\alpha$ for some constants $C > 0, \alpha > 0$.*

Proof. Let s be the transition time of J into I . Take an arbitrary $x \in D(J) \cap J$ and let r be the first return time of x into J . By Lemma 1, $r \geq s$. By the Koebe principle and non-flatness of the critical point, it suffices to show that $U := f^s(\mathcal{L}_x(J))$ is well inside I . If $r = s$, then $U \subset J$ is τ -well inside I , so we are done. Otherwise, $U \subset \mathcal{L}_{f^s(x)}(J)$, and by Theorem 3, $\mathcal{L}_{f^s(x)}(J)$ is τ' -well inside I , where $\tau' > 0$ depends only on τ . \square

Lemma 3. *There exists a universal constant $\rho_0 > 0$ such that if $I \ni c$ is a small nice interval and $J \neq I^1$ is a child of I , then J is ρ_0 -well inside I .*

Proof. Let s be the return time of c to I and let $m \geq 1$ be such that $f^s(c) \in I^{m-1} \setminus I^m$. Note that $J \subset I^m$ and $f^s(J) \subset I^{m-1} \setminus I^m$.

Let $\rho > 0$ be the constant appearing in Theorem 2. If I^m is ρ -well inside I^{m-1} , then J is ρ -well inside I , and we are done. So assume that I^m is not ρ -well inside I^{m-1} . Then $R_{I^{m-1}} : I^m \rightarrow I^{m-1}$ is a high return, and $f^{s-1}|f(I^m)$ has uniformly bounded distortion. Since $f^s(I^m)$ is definitely large than $f^s(J)$, it follows that $|f(J)|/|f(I^m)|$ is bounded away from one, hence J is uniformly well inside $I^m \subset I$. \square

3.2. Central cascade. By a *central cascade*, we mean a sequence of symmetric nice intervals

$$T \supset T^1 \supset \cdots \supset T^m, \quad (m \geq 1)$$

which contain c such that

- T^{i+1} is the central return domain of T^i , for each $0 \leq i < m$;
- the first return time of c to T, T^1, \dots, T^{m-1} are all the same.

So R_{T^i} are central for all $0 \leq i \leq m-2$. A central cascade is called *maximal* if $R_T(c) \notin T^m$.

A nice interval I is called τ -*nice*, if each return domain of I is τ -well inside I .

Proposition 2. *Let $T = T^0 \ni c$ be a small symmetric nice interval and let $T^0 \supset T^1 \supset T^2 \supset \cdots \supset T^m$ be a maximal central cascade. Assume that T^1 is τ -well inside T^0 . Let $i \in \{1, 2, \dots, m\}$ and let $J_1 \supsetneq J_2 \supsetneq \cdots$ be all the children of T^i . Then there exist constants $C > 0$ and $0 < \lambda < \lambda_0 < 1$, depending only on τ , such that*

1. *for each $k = 1, 2, \dots$, we have $|J_k| \leq \lambda^{k-1}|T^i|$;*
2. *for each $k \geq 2$, J_k is $C\lambda_0^{-k}$ -nice.*

To prove this proposition, let us first introduce some notation. For $y \in D(T^0)$, let $r(y)$ denote the first entry time of y into T^0 , and let $s = r(c)$, so $R_{T^0}|T^1 = f^s|T^1$. Let

$$E(T) = \bigcup_{i=0}^{m-1} \{x \in T^i \setminus T^{i+1} : R_T^i(x) \in D(T)\},$$

and for each $x \in E(T) \cap (T^i \setminus T^{i+1})$, let

$$t(x) = is + r(f^{is}(x)).$$

Moreover, let $F = F_T : E(T) \rightarrow T$ be defined as

$$F(x) = f^{t(x)}(x).$$

Clearly, $t(x)$ is constant in each component J of $E(T)$.

We shall also need the following notations:

- $Q = f^{-s}(T^m) \cap T^m$;
- V is the component of $f^{-s}(E(T))$ which contains c ;
- $X = f^{-s}(E(T)) \cap (T^m \setminus (Q \cup V))$.

Lemma 4. (i) *The map F maps each component J of $E(T)$ diffeomorphically onto T .*
(ii) *For each $x \in D(T^m) \setminus (Q \cup V)$, if k is the entry time of x to T^m , then f^k maps a neighborhood $W(x)$ of $\mathcal{L}_x(T^m)$ diffeomorphically onto T . Moreover, if, in addition, $x \in X$ then $W(x) \subset X$.*

Proof. We first prove the statement (i). If J is a component of $E(T)$ in $T^0 \setminus T^1$, then J is a non-central return domain and $F = R_T$, so F maps J diffeomorphically onto T . Now let J be a component of $E(T)$ in $T^i \setminus T^{i+1}$ for some $1 \leq i < m$. Since f^{is} maps a component of $T^i \setminus T^{i+1}$ diffeomorphically onto a component of $T^0 \setminus T^1$, $f^{is} : J \rightarrow J' := f^{is}(J)$ is a diffeomorphism and J' is a component of $E(T)$ in $T^0 \setminus T^1$. Since $t|_J = t|_{J'} + is$, $F|_J = R_T|_{J'} \circ f^{is}|_J$ maps J diffeomorphically onto T .

Let us prove the statement (ii). Let us distinguish a few cases.

Case 1. $x \in T \setminus T^m$. In this case, $f^k|_{\mathcal{L}_x(T^m)}$ can be written as an iterate of F , so the statement follows from (i). Note that $W(x) \subset E(T)$.

Case 2. $x \notin T$. Let $k' \leq k$ be the first entry time of x to T . Then $f^{k'} : \mathcal{L}_x(T) \rightarrow T$ is a diffeomorphism. So the statement holds if $k' = k$. If $k' < k$, then $f^{k'}(\mathcal{L}_x(T^m)) = \mathcal{L}_{f^{k'}(x)}(T^m)$ and we are reduced to Case 1.

Case 3. $x \in D(T^m) \cap X$. Then $k > s$ and $x' = f^s(x) \in D(T^m) \cap (T^{m-1} \setminus T^m)$. Let $W_0(x)$ and $W_0(x')$ denote the component of X which contains x and x' respectively. By definition of X , $f^s : W_0(x) \rightarrow W_0(x')$ is a diffeomorphism. So we are reduced to Case 1 again. \square

A nice interval I is called τ -non-central nice if all its return domains, except possibly the one containing c , are τ -well inside I . The following is an immediate consequence of Lemma 4.

Lemma 5. *Assume that T^1 is τ -well inside T^0 . Then for each $1 \leq i < m$, T^i is a τ' -non-central-nice interval, where τ' depends only on τ .*

Proof. Note that for each return domain U of T^i , $U \neq T^{i+1}$, the first return map $R_{T^i}|_U$ can be written in the form $F^n|_U$ for some $n \geq 1$. By Lemma 4, it follows that $f^k : U \rightarrow T^i$ extends to a diffeomorphism $f^k : \widehat{U} \rightarrow T^0$ and $\widehat{U} \subset T^i \setminus T^{i+1}$, where $k \geq 1$ is the first return time of J into T^i . Since

$T^i \subset T^1$ is τ -well inside T , by the Koebe principle, U is well inside \widehat{U} , hence well-inside T^i . \square

Lemma 6. *For any $\tau > 0$, there exists $\tau' > 0$ such that if $T \ni c$ is a small τ -non-central-nice interval and $J_2 \subset J_1$ are children of T , then J_2 is τ' -well inside J_1 .*

Proof. Let $s_1 < s_2$ be the transition time of J_1, J_2 to T respectively. Let s be the maximal integer such that $s_1 \leq s < s_2$ and $x := f^s(c) \in T$. Then $s_2 - s$ is the return time of x into T and $\mathcal{L}_x(T)$ does not contain c . By assumption, $\mathcal{L}_x(T)$ is τ -well inside T , so by Theorem 3, the component of $f^{-(s-s_1)}(\mathcal{L}_x(T))$ containing $f^{s_1}(c)$ is τ' -well inside T , where $\tau' > 0$ is a constant. By Theorem 1 and non-flatness of the critical point, J_2 is well inside J_1 . \square

These lemmas imply Proposition 2 immediately unless

$$(6) \quad (1 + 2\rho)T^m \supset T^{m-1}.$$

To deal with the case when (6) holds, we need the following three lemmas.

Assume (6). Then by Theorem 2, $R_{T^{m-1}} : T^m \rightarrow T^{m-1}$ is high, so Q consists of two intervals, each of which is mapped diffeomorphically onto T^m by f^s . Let Q_+, Q_- denote the components of Q such that $f^s|_{Q_+}$ is monotone increasing. Let b be the unique fixed point of $f^s|_{Q_-}$, let $\hat{b} = (f^s|_{Q_+})^{-1}(b)$ and let $B = (b, \hat{b})$.

Lemma 7. *There exist universal constants $K > 1$ and $\sigma > 0$ such that the following hold:*

- (i) *For any $x, y \in T^m$, $|Df^{s-1}(f(x))| \leq K|Df^{s-1}(f(y))|$;*
- (ii) *$|(f^s)'(x)| \leq K$ holds for all $x \in T^m$;*
- (iii) *for any measurable $A \subset T^m$, $\frac{|A|}{|T^m|} \leq K \left(\frac{|f^s(A)|}{|T^m|} \right)^{1/\ell}$;*
- (iv) *f^s maps a neighborhood Z of Q_+ diffeomorphically onto its image and $\widehat{Z} := f^s(Z) \supset Z \cup T^m \cup (1 + 2\sigma)Q_+$.*
- (v) *$\sigma|T^m| \leq |B| \leq (1 - \sigma)|T^m|$.*

Proof. By Theorem 2, f^{s-1} maps a neighborhood G_1 of $f(T^m)$ diffeomorphically onto $G := (1 + 2\rho)T^{m-1}$. By the real Koebe principle, there exists $K > 1$ such that (i) holds. For $x \in T^m$, we have

$$|(f^s)'(x)| = |f'(x)| |(f^{s-1})'(f(x))| \leq K \frac{|f^s(T^m)|}{|f(T^m)|} |f'(x)|.$$

Since $|f^s(T^m)| \leq |T^{m-1}| \leq (1 + 2\rho)|T^m|$, by the non-flatness, it follows that the statement (ii) holds by redefining the constant K . The statement (iii) follows from (i) in a similar way. For (iv) and (v), assume for definiteness that Q_+ lies to the left of c . Let $G_0 = f^{-1}(G_1)$ and let $Z = (u, c)$ be the left component of $G_0 \setminus \{c\}$. Then f^s maps Z diffeomorphically onto its image and $\widehat{Z} := f^s(Z) \supset T^m$. Since $f^s(u)$ is the left endpoint of G , we have $f^s(Z) \supset (1 + 2\sigma)Q_+$, where $\sigma = \min(1, \rho)$. If $Z \not\subset f^s(Z)$, then

f^s would map $Z \setminus I^m$ into itself and hence f^s would have an attracting fixed point, which is not possible. Thus $f^s(Z) \supset Z$. The statement (iv) is proved. The statement (iv) follows from (ii), since $f^s(Q_- \cap \overline{B}) \supset Q_- \setminus B$ and $f^s(Q_- \setminus B) \supset Q_- \cap B$. \square

Lemma 8. *Assume that $(1+2\rho)T^m \not\subset T^{m-1}$. Then there exists a universal constant $\theta \in (0, 1)$ such that if P is an interval such that $f^{js}(P) \subset Q$ for $j = 0, 1, \dots, N-1$, then*

$$|P| \leq \theta^N |T^m|.$$

Proof. Let

$$\mathcal{P}_n = \{x \in T^m : f^{is}(x) \in Q \text{ for } 0 \leq i < n\}$$

and

$$\mathcal{P}_n^* = \{x \in \mathcal{P}_n : f^{ns}(x) \in T^m \setminus \{b, \hat{b}\}\}.$$

Note that each component of \mathcal{P}_n is the union of three intervals of \mathcal{P}_n^* , up to two points (corresponding to preimages of b and \hat{b}). As each component of \mathcal{P}_n , $n \geq 1$, is at most of length $|T^m|/2$, it suffices to show there exist universal constants $C_* > 0$ and $\theta_* \in (0, 1)$ such that for each component P_n^* of \mathcal{P}_n^* we have

$$(7) \quad |P_n^*| \leq C_* \theta_*^n |T^m|.$$

Let $Q_{+,1} = Q_+$ and $Q_{-,1} = Q_-$, and for each $j > 1$, let $Q_{+,j} = (f^s|_{Q_+})^{-1}(Q_{+,j-1})$ and $Q_{-,j} = (f^s|_{Q_-})^{-1}(Q_{-,j-1})$. Then $Q_{+,j}$ are symmetric to $Q_{-,j}$ with respect to c .

Claim 1. There exists a universal constant $\theta_1 \in (0, 1)$ such that

$$|Q_{+,j}| = |Q_{-,j}| \leq \theta_1^j |T^m|.$$

Indeed, by (iv) of Lemma 7, for each $j \geq 1$, f^{js} maps a neighborhood of $Q_{+,j}$ diffeomorphically onto \hat{Z} . Since $f^{js}(Q_{+,j+1}) = Q_+$, and $f^{js}(Q_{+,j}) = T^m$, it follows by the Koebe principle that $|Q_{+,j+1}|/|Q_{+,j}|$ is uniformly bounded away from 1. The claim follows.

Let

$$\mathcal{B}_n = \{x \in T^m : f^{js}(x) \in Q \text{ for } 0 \leq j < n, f^{ns}(x) \in B\} \subset \mathcal{P}_n^*.$$

For each component B_n of \mathcal{B}_n , f^{ns} maps a neighborhood of B_n diffeomorphically onto T^m . By (v) of Lemma 7, B is uniformly well inside T^m . By the Koebe principle, there exists a universal constant $K_1 > 1$ such that

$$(8) \quad \sup_{x,y \in B_n} \frac{|(f^{ns})'(x)|}{|(f^{ns})'(y)|} \leq K_1.$$

Claim 2. There exists a universal constant $\theta_2 \in (0, 1)$ such that for each component B_n of \mathcal{B}_n , $n = 0, 1, \dots$, we have

$$(9) \quad |B_n| \leq \theta_2^n |T^m|.$$

To prove this claim, let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n \subset D(B)$. For each $x \in \mathcal{B}$, the first entry time of x into B is of the form $k(x)s$, where $k(x) \geq 1$ is an

integer. For $x \in \mathcal{B} \setminus B$, we have $f^{js}(x) \in Q_+$ for $1 \leq j < k(x)$, so $\mathcal{L}_x(B) \subset Q_{+,k(x)} \cup Q_{-,k(x)}$. Thus by Claim 1, we have

$$(10) \quad |\mathcal{L}_x(B)| \leq \theta_1^{k(x)} |T^m| \text{ holds for all } x \in \mathcal{B} \setminus B.$$

Let us now show that there exist a universal constant $\theta_3 \in (0, 1)$ such that

$$(11) \quad |\mathcal{L}_x(B)| \leq \theta_3^{k(x)} |B| \text{ holds for all } x \in \mathcal{B} \cap B.$$

Indeed, $\mathcal{L}_x(B)$ lies in a component of $B \setminus \{c\}$, so

$$(12) \quad |\mathcal{L}_x(B)| \leq |B|/2.$$

In particular, if $k(x) = 1$, then (11) holds with $\theta_3 = 1/2$. If $k(x) > 1$, then $f^s(x) \in \mathcal{B} \setminus B$ and $f^s(\mathcal{L}_x(B)) = \mathcal{L}_{f^s(x)}(B)$. So by (10) and part (iii) and (v) of Lemma 7, we have

$$\frac{|\mathcal{L}_x(B)|}{|B|} \leq \sigma_2^{-1} \frac{|\mathcal{L}_x(B)|}{|T^m|} \leq \frac{K_2}{\sigma_2} \theta_1^{(k(x)-1)/\ell}.$$

Together with (12), it follows that (11) holds for a suitable choice of θ_3 .

Now let us prove (9). Take a component B_n of \mathcal{B}_n , $n \geq 1$. Let $1 \leq n_1 < n_2 < \dots < n_k = n$ be all the positive integers such that $f^{n_i s}(B_n) \subset B$ and let B_{n_i} be the component of \mathcal{B}_{n_i} which contains B_n . Then $B_{n_1} \supset B_{n_2} \supset \dots \supset B_{n_k} = B_n$. For each $1 \leq i < k$, $Y_i := f^{n_i s}(B_{n_{i+1}})$ is a component of \mathcal{B} with entry time $(n_{i+1} - n_i)s$. By (8),

$$\frac{|B_{n_{i+1}}|}{|B_{n_i}|} \leq \frac{K_1 |Y_i|}{|B| + (K_1 - 1) |Y_i|}.$$

Thus by (11), there exists $\theta_4 \in (0, 1)$ such that $|B_{n_{i+1}}| \leq \theta_4^{n_{i+1} - n_i} |B_{n_i}|$. So $|B_n| \leq \theta_4^{n - n_1} |B_{n_1}|$. Let $\theta_2 = \max(\theta_1, \theta_3, \theta_4)$. By (10) and (11), $|B_{n_1}| \leq \theta_2^{n_1} |T^m|$. Thus (9) holds.

Now let us complete the proof. Let P_n^* be a component of \mathcal{P}_n^* . We may assume $P_n^* \notin \mathcal{B}_n$ for otherwise, Claim 2 applies. Write $P_i^* = f^{(n-i)s}(P_n^*)$. Let n_0 be maximal in $\{0, 1, \dots, n\}$ such that $P_i^* \cap B = \emptyset$ for all $0 \leq i \leq n_0$. Since $f^s(Q \setminus B) \cap (Q_- \setminus B) = \emptyset$, we have $f^{is}(P_{n_0}^*) \subset Q_+$ for all $1 \leq i < n_0$. So $P_{n_0}^* \subset Q_{+,n_0}$ or $P_{n_0}^* \subset Q_{-,n_0}$. By Claim 1, we have $|P_{n_0}^*| \leq \theta_1^{n_0} |T^m|$. If $n_0 = n$ then we are done again. Assume $n_0 < n$. Then $P_{n_0+1}^* \subset B$. By part (iii) and (v) of Lemma 7, $|P_{n_0+1}^*| \leq C'_1 \theta_1^{n_0+1} |B|$ holds for some universal constants $C'_1 > 0$ and $\theta'_1 \in (0, 1)$. Let B_{n-n_0-1} be the component of \mathcal{B}_{n-n_0-1} which contains P_n^* . By (8), we have

$$|P_n^*| \leq K_1 |B_{n-n_0-1}| \frac{|P_{n_0+1}^*|}{|B|}.$$

By Claim 2, the inequality (7) follows. \square

Lemma 9. Assume $(1 + 2\rho)T^m \not\subset T^{m-1}$. Let $y \in V$ and let t be the first return time of y to T^m . Assume that $f^{js}(f^t(y)) \in Q$ for all $j = 0, 1, \dots, n-1$ and let H be the component of $f^{-ns-t}(T^m)$ which contains y . Then

$$|H| \leq \theta_0^n |T^m|,$$

where $\theta_0 \in (0, 1)$ is a constant depending on τ .

Proof. Let $\delta \in (0, \rho)$ be such that $|T^{m-1}| = (1 + 2\delta)|T^m|$. Since $f^s(V) \subset T^{m-1} \setminus T^m$ we have $|f^s(V)| \leq \delta|T^m|$. By part (i) of Lemma 7, $|fV| \leq K\delta|f(T^m \setminus V)|$. By non-flatness, there exist universal constants $K_1 > 1$ and $\eta_1 \in (0, 1)$ such that

$$\frac{|V|}{|T^m|} \leq \min(\eta_1, K_1\delta^{1/\ell}) =: \eta.$$

Since $H \subset V$, we obtain

$$(13) \quad |H| \leq \eta|T^m|.$$

Take $\gamma \in (0, 1)$ such that $K_1^\gamma \eta_1^{1-\gamma} = 1$.

Case 1. $\delta < \theta^{n/2}$. Then

$$\eta \leq (K_1\delta^{1/\ell})^\gamma \eta_1^{1-\gamma} \leq \theta^{n\gamma/(2\ell)},$$

so we are done in this case.

Case 2. $\delta \geq \theta^{n/2}$. By Lemma 8, $|f^t(H)| \leq \theta^n|T^m|$. So $f^t(H)$ is $\theta^{-n/2}$ -well inside T^{m-1} . By Lemma 4 (ii), f^{t-1} maps an interval $W \ni f(y)$ diffeomorphically onto T^{m-1} . Let W_0 be the component of $f^{-1}(W)$ which contains y . Then $W_0 \subset T^m$. By the Koebe principle and non-flatness, we obtain that

$$|H| \leq C\theta^{n/2\ell}|W_0| \leq C\theta^{n/2\ell}|T^m|,$$

where $C = C(\theta)$ is a constant. Together with (13), this implies the statement. \square

Proof of Proposition 2. The second statement follows from the first by Lemma 2. In the following we shall prove the first statement.

By Lemmas 5 and 6, the first statement holds in the case $1 \leq i < m$. In the following, we shall estimate the size of children of T^m .

If $(1 + 2\rho)T^m \subset T^{m-1}$, then by Lemma 2, T^m is ρ' -nice for some $\rho' > 0$ and so we are done again by Lemma 6. We assume from now on that $(1 + 2\rho)T^m \not\subset T^{m-1}$, so that Lemmas 7, 8 and 9 apply.

For each $i = 1, 2, \dots$, let S_i denote a transition time from J_i to T^m . By definition, f^{S_i-1} maps an interval \widehat{J}_i which contains $f(J_i)$ diffeomorphically onto T^m . Let $i(1) = \inf\{i \geq 1 : f^{S_i}(c) \notin Q\}$, and define inductively,

$$i(j+1) = \inf\{i > i(j) : f^{S_i}(c) \notin Q\}.$$

For $i \in \{1, 2, \dots, i(1)\}$, applying Lemma 9 to $y = c$ and $n = i$, we obtain that $|J_i| \leq \theta_0^i|T^m|$.

It remains to show that for each $i(j) < i \leq i(j+1)$, $|J_i| \leq \theta_2^{i-i(j)}|J_{i(j)}|$ holds for some constant $\theta_2 = \theta_2(\tau) \in (0, 1)$. To this end, let $y := f^{S_{i(j)}}(c)$ and we distinguish two cases.

Case 1. $y \in X$. Then $k := S_{i(j)+1} - S_{i(j)}$ is the first return time of y into T^m . By Lemma 4, f^k maps an interval $W(y)$ with $y \in W(y) \subset X$ diffeomorphically onto $T \supset (1 + 2\tau)T^m$. By Lemma 8, $|f^{S_{i(j)+1}}(J_i)| \leq \theta^{i-i(j)-1}|T^m|$, so $f^{S_{i(j)+1}}(J_i)$ is $\tau\theta^{i(j)-i+1}$ -well inside T . By the Koebe principle, $f^{S_{i(j)}}(J_i)$ is $\tau'\theta^{i(j)-i}$ -well inside $W(y)$ for some constants $\tau' > 0$

and $\theta' \in (0, 1)$. Applying the Koebe principle again to the diffeomorphism $f^{S_{i(j)}-1} : \widehat{J}_{i(j)} \rightarrow T^m$ and using the non-flatness of critical point, we obtain the desired estimate.

Case 2. $y \in V$. In this case, applying Lemma 9 to y and $n = i - i(j)$, we obtain that $|f^{S_{i(j)}}(J_i)| \leq \theta_0^{i-i_j} |T^m|$. So $f^{S_{i(j)}}(J_i)$ is $C\tilde{\theta}_0^{i_j-i}$ -well inside T^m for some constants $C > 0$ and $\tilde{\theta}_0 \in (0, 1)$. Applying the Koebe principle again to the diffeomorphism $f^{S_{i(j)}-1} : \widehat{J}_{i(j)} \rightarrow T^m$ and the non-flatness of critical point, we obtain the desired estimate. \square

4. PULL BACK OF EMPTY SPACE

In this section, we will assume that f is non-renormalizable and that $\omega(c)$ is a wild attractor. The latter implies that the set $\Lambda(g)$ defined later has positive Lebesgue measure. For each small nice interval $T \ni c$, we shall first define a parameter $\xi(T)$ which measures the relative size of the complement of $\Lambda(g)$ (“empty space”) in T . Then we shall study the distortion of this parameter under pull back by f . The main results are Propositions 3 and 4.

Fix a symmetric nice interval $I \ni c$ such that ∂I contains no periodic points of f . Let $\mathbf{D}(I)$ is the union of return domains of I which have non-empty intersection with $\omega(c)$ and let $g = R_I|_{\mathbf{D}(I)}$. Since $\omega(c)$ is minimal, $\mathbf{D}(I)$ has only finitely many components, each of which is compactly contained in I . So $I \setminus \mathbf{D}(I)$ has non-empty interior. The non-escaping set of g is defined by

$$\Lambda(g) = \{x \in \mathbf{D}(I) : g^n(x) \in \mathbf{D}(I) \text{ for all } n \geq 0\}.$$

For $T \subset I$, let

$$\Lambda_g(T) = \{x \in \mathbf{D}(I) : \exists k \geq 1, g^k(x) \text{ is well defined and contained in } T\}.$$

For an interval $Y \subset [0, 1]$ and a set $Y' \subset Y$, we will define a number $\lambda(Y'|Y)$ to measure how much the subset Y' occupies in Y . Let \mathcal{F}_Y be the set of diffeomorphisms of the form $f^s : J \rightarrow Y$. Define

$$\lambda(Y'|Y) = \sup_{\phi \in \mathcal{F}_Y} \frac{|\phi^{-1}(Y')|}{|\phi^{-1}(Y)|},$$

and

$$\xi(Y'|Y) = 1 - \lambda(Y'|Y).$$

For a nice interval $T \subset I$ with $T \ni c$, define

$$\xi(T) = \xi(\Lambda_g(T) \cap T|T).$$

Remark. For each small nice interval T , $\xi(T) > 0$. Indeed, f is topologically transitive on $[f(c), f^2(c)] \supset T$, so $T \setminus \Lambda_g(T)$ has non-empty interior. Moreover, since $\omega(c)$ is minimal, $\partial T \cap \omega(c) = \emptyset$, so there exists $\delta = \delta(T) > 0$ such that each diffeomorphism $f^s : J \rightarrow T$ extends to a diffeomorphism onto

the δ -neighborhood of T . By the Koebe principle, there exists a constant $C = C(T) > 0$ such that

$$\frac{|(f^s|J)^{-1}(T \setminus \Lambda_g(T))|}{|J|} \geq C \frac{|T \setminus \Lambda_g(T)|}{|T|} > 0.$$

Lemma 10. *Suppose that f has a wild attractor. Let $T_n \ni c$ be a sequence of nice intervals such that $|T_n| \rightarrow 0$ as $n \rightarrow \infty$. Then $\xi(T_n) \rightarrow 0$.*

Proof. Assume by contradiction that there exists a sequence of nice intervals $T_n \ni c$ and a constant $\lambda > 0$ such that $|T_n| \rightarrow 0$ and $\xi(T_n) \geq \lambda$, $n = 1, 2, \dots$. Since f has a wild attractor, the non-escaping set $\Lambda(g)$ has positive Lebesgue measure. Let $X = \{x \in \Lambda(g) : \omega(x) \ni c\}$. Then by Mâné's theorem [13], $|X| = |\Lambda(g)| > 0$. Let n_0 be large such that $|X \setminus T_{n_0}| > 0$ and let $x \in X \setminus T_{n_0}$ be a Lebesgue density point of X . For each $n \geq n_0$, let s_n be the first entry time of x under f to T_n and let $J_n = \mathcal{L}_x(T_n)$. Then $f^{s_n} : J_n \rightarrow T_n$ is a diffeomorphism. Since $f^{s_n}(X) \subset \Lambda_g(T_n)$, we have

$$\frac{|J_n \cap X|}{|J_n|} \leq \frac{|J_n \cap f^{-s_n}(\Lambda_g(T_n))|}{|J_n|} \leq \lambda(T_n \cap \Lambda_g(T_n)|T_n) \leq 1 - \lambda.$$

Since f has no wandering interval [15], $|J_n| \rightarrow 0$. This contradicts with the assumption that x is a Lebesgue density point of X . \square

The following lemma is an improvement of Lemma 4.11 of [11].

Lemma 11. *Let T be small interval. Let U_1, U_2, \dots and W_1, W_2, \dots, W_k be pairwise disjoint subintervals of T and $Y \subset (\cup_i U_i) \cup (\cup_{i=1}^k W_i)$. Assume that*

- for each i , $\lambda(Y \cap U_i|U_i) \leq \lambda$;
- for each $1 \leq j \leq k$, W_j is τ -well inside T .

Then there exists $\varepsilon = \varepsilon(k, \tau) \in (0, 1)$ such that

$$(14) \quad 1 - \lambda(Y|T) \geq (1 - \varepsilon)(1 - \lambda).$$

Moreover, for a fixed k , $\varepsilon(k, \tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

Proof. Let $\mu_k(\tau) = (1 + \tau^{-1})^{-k}$. We first prove that

$$(15) \quad \sum_{j=1}^k |W_j| \leq (1 - \mu_k(\tau))|T|.$$

Without loss of generality, we may assume that W_1, W_2, \dots, W_k lie from left to right in T . Let $\delta_j = |W_j|/|T|$ and let $\rho = 1 - \sum_{j=1}^k \delta_j$. Since the left component of $T \setminus W_j$ has length at least $\tau|W_j|$, we obtain

$$\rho \geq \tau\delta_1,$$

and for each $j = 2, 3, \dots, k$,

$$\rho + \sum_{j'=1}^{j-1} \delta_{j'} \geq \tau\delta_j.$$

By induction, it follows that for each $j = 1, 2, \dots, k$,

$$\delta_j \leq \frac{\rho}{\tau} (1 + \tau^{-1})^{j-1}.$$

Since $\rho + \delta_1 + \dots + \delta_k = 1$, this implies that

$$\rho \geq (1 + \tau^{-1})^{-k} = \mu_k(\tau),$$

hence

$$\delta_1 + \delta_2 + \dots + \delta_k \leq 1 - \mu_k(\tau).$$

The inequality (15) is proved.

Now let $\tau' = 0.9 \frac{\tau^2}{1+2\tau}$ and $\varepsilon(k, \tau) = 1 - \mu_k(\tau')$. Clearly, for a fixed k , $\varepsilon(k, \tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$. It remains to show that (14) holds with $\varepsilon = \varepsilon(k, \tau)$. To this end, take an arbitrary diffeomorphism $\phi : T' \rightarrow T$ from the class \mathcal{F}_T . Let U'_i, W'_j, Y' be the pre-images of U_i, W_j, Y under ϕ respectively. By the Koebe principle, W'_j is τ' -well inside T' . Therefore as above, we obtain

$$\sum_{i=1}^k |W'_j| \leq (1 - \mu_k(\tau')) |T'|.$$

For each $i \geq 1$,

$$|Y' \cap U'_i| \leq \lambda(Y \cap U_i | U_i) |U'_i| \leq \lambda |U'_i|.$$

Putting $U' = \bigcup U'_i$ and $W' = \bigcup_j W'_j$, we have $Y' \subset U' \cup W'$. Thus

$$\begin{aligned} |T' \setminus Y'| &\geq \sum_i |U'_i \setminus Y'| + |T' \setminus (U' \cup W')| \\ &\geq (1 - \lambda)(|U'| + |T' \setminus (U' \cup W')|) \\ &= (1 - \lambda)|T' \setminus W'| \\ &\geq (1 - \lambda)\mu_k(\tau')|T'|. \end{aligned}$$

The inequality (14) follows. \square

Lemma 12. *Let T be a small interval, let T' be a unimodal pull back of T by f^s , and let $Y' \subset T'$. Assume that $Y := f^s(Y')$ is covered by subintervals U_i of T , $i = 0, 1, 2, \dots$, such that*

- for each $i \geq 0$, U_i is τ -well inside T ;
- for each $i \geq 1$, $\lambda(Y | U_i) \leq \lambda$.

Then

$$(16) \quad 1 - \lambda(Y' | T') \geq (1 - \varepsilon(\tau))(1 - \lambda).$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1/\ell})$ as $\tau \rightarrow \infty$.

Proof. For each $i \geq 1$, $f^{-s}(U_i) \cap T'$ has at most two components and if $f^s(c) \in U_i$ then $f^{-s}(U_i) \cap T'$ is an interval. Let U'_j be the components of $f^{-s}(\bigcup_{i=0}^{\infty} U_i)$ such that for each $j \geq 4$, $f^s|_{U'_j}$ is a diffeomorphism onto U_i for some $i \geq 1$, hence

$$\lambda(Y' | U'_j) \leq \lambda(Y | U_i) \leq \lambda.$$

By the Koebe principle, each U'_j is τ' -well inside T' , where τ' is a constant depending only on τ and $\tau' = O(\tau^{1/\ell})$ as $\tau \rightarrow \infty$. Thus by Lemma 11, the statement follows. \square

Lemma 13. *Let $T \subset I$ be a nice interval that contains c such that T^1 is τ -well inside T and let $K \subset T \setminus T^1$ be a component of $\Lambda_g(T)$. Then there exists $\varepsilon = \varepsilon(\tau) > 0$, such that*

$$(17) \quad \lambda(K \cap \Lambda_g(T^1) | K) \leq \frac{\varepsilon}{\xi(T) + \varepsilon}.$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

Proof. Let $U_0 = T \setminus \Lambda_g(T)$, $V_0 = (\Lambda_g(T) \cap T) \setminus T^1$ and $W_0 = T^1$. Moreover, for each $k \geq 1$, inductively define

$$\begin{aligned} U_k &= \{x \in V_{k-1} : R_T^k(x) \in U_0\}, \\ V_k &= \{x \in V_{k-1} : R_T^k(x) \in V_0\}, \\ W_k &= \{x \in V_{k-1} : R_T^k(x) \in W_0\}. \end{aligned}$$

Since T^1 is τ -well inside T , by the Koebe principle, there exists $\varepsilon = \varepsilon(\tau) > 0$ such that for each $\psi \in \mathcal{F}_T$, we have

$$|\psi^{-1}(T^1)| \leq \varepsilon |\psi^{-1}(T)|,$$

where $\varepsilon = \varepsilon(\tau) = (1 + 2\theta)^{-1}$ and $\theta = 0.9\tau^2/(1 + 2\tau)$. So $\varepsilon(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

For each component J of V_{k-1} , $R_T^k|J$ is a diffeomorphism onto T , so for each $\phi \in \mathcal{F}_J$, we have $R_T^k|J \circ \phi \in \mathcal{F}_T$. Therefore,

$$|\phi^{-1}(U_k \cap J)| \geq \xi(T) |\phi^{-1}(J)|,$$

and

$$|\phi^{-1}(W_k \cap J)| \leq \varepsilon |\phi^{-1}(J)|.$$

So

$$\frac{|\phi^{-1}(U_k \cap J)|}{|\phi^{-1}(W_k \cap J)|} \geq \frac{\xi(T)}{\varepsilon}.$$

By Mañé's Theorem [13], $\bigcap_k V_k$ has measure zero. It follows that

$$\frac{|\phi^{-1}(T \cap W)|}{|\phi^{-1}(T)|} \leq \frac{\varepsilon}{\xi(T) + \varepsilon},$$

where

$$W := \bigcup_{k=0}^{\infty} W_k.$$

Thus,

$$\lambda(T \cap W | T) \leq \frac{\varepsilon}{\xi(T) + \varepsilon}.$$

For each component K of V_0 , since the first return map R_T maps $K \cap \Lambda_g(T^1)$ onto W , we have $\lambda(K \cap W | K) \leq \lambda(T \cap W | T)$. The lemma follows. \square

Proposition 3. *For any $\tau > 0$, there exists $\varepsilon = \varepsilon(\tau) \in (0, 1)$, such that for any τ -nice interval $T \subset I$ with $T \ni c$ and any child J of T ,*

$$\frac{\xi(J)}{\xi(T)} \geq \frac{1 - \varepsilon}{\xi(T) + \varepsilon}.$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1/\ell})$ as $\tau \rightarrow \infty$.

Proof. Let s be a transition time of J to T . By Lemma 1 $Y := f^s(\Lambda_g(J) \cap J) \subset \Lambda_g(T^1) \cup T^1$. Let U_0, U_1, \dots be the components of $\Lambda_g(T) \cap T$ such that $U_0 \ni c$. Then for all $i \geq 0$, U_i is τ -well inside T . By Lemma 13, for each $i \geq 1$, $\xi(Y|U_i) \geq \xi(T)/(\xi(T) + \varepsilon_1)$, where $\varepsilon_1 = \varepsilon_1(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$. By Lemma 12, the statement follows. \square

The previous proposition says that the empty space of a unimodal pull back does not decrease too much. Now we will show that the central cascade does not influence the empty space too much as well.

Definition 4.1. Given a maximal central cascade $T \supset T^1 \supset \dots \supset T^m$, an inheritor of T is, by definition, a child J of $T^{m'}$ for some $0 \leq m' \leq m$ such that $J \subsetneq T^m$.

Proposition 4. *Let $T \supset T^1 \supset \dots \supset T^m$ be a maximal central cascade, where $T \ni c$ is a small symmetric τ -nice interval. Then there is a constant $C = C(\tau) > 0$ such that for each inheritor J of T , we have*

$$\xi(J) \geq C\xi(T).$$

Proof. Let us first prove the proposition under the following assumption:

(*) each component of $D(T) \cap (T \setminus T^1)$ is τ -well inside $T \setminus T^1$.

Let E_T, F_T, V, Q , and X be as defined in § 3.2. Let V_0 be the component of $T^m \setminus \overline{Q}$ which contains c . Let $Q' = Q \cap D(X \cup V_0)$. Note that $X \cup V_0$ is a nice set and for each component K of Q' , the first entry time of K into $X \cup V_0$ is of the form ns and f^{ns} maps a neighborhood of K in Q diffeomorphically onto T^m .

Claim 1. There exists a constant $\tau' > 0$ such that

- (1a) for each $0 \leq i < m$, each component of $E(T) \cap (T^i \setminus T^{i+1})$ is τ' -well inside T^i ;
- (1b) V is τ' -well inside V_0 ;
- (1c) each component of X is τ' -well inside T^m ;
- (1d) each component of Q' is τ' -well inside T^m .

Proof of Claim 1. (1a). By assumption, if K is a component of $E(T) \cap (T \setminus T^1)$, then K is τ -well inside $T \setminus T^1 \subset T$. For each $1 \leq i < m$, f^{is} maps each component of $T^i \setminus T^{i+1}$ diffeomorphically onto $T \setminus T^1$ and for each component K of $E(T) \cap (T^i \setminus T^{i+1})$, $K' = f^{is}(K)$ is a component of $E(T) \cap (T \setminus T^1)$. The statement follows by the Koebe principle.

(1b). Note that V_0 is a unimodal pull back of the component of $T^{m-1} \setminus T^m$ which contains $f^s(c)$. Since $f^s(V) \subset E(T)$, by (1a), $f^s(V)$ is well inside

$T^{m-1} \setminus T^m$. Thus the statement follows by the Koebe principle and non-flatness of critical point. (We need to redefine the constant τ' .)

(1c). It also follows from (1a) by the Koebe principle and non-flatness of critical point.

(1d) follows from (1b) and (1c) and the observation on the components Q' by the Koebe principle. \square

Claim 2. There exists a constant $C_0 > 0$ such that

(2a) for each component K of $E(T) \cap (T \setminus T^1)$, we have

$$\xi(\Lambda_g(T^1) \cap K|K) \geq C_0 \xi(T);$$

(2b) for each $1 \leq i < m$ and each component K of $E(T) \cap (T^i \setminus T^{i+1})$, we have

$$\xi(\Lambda_g(T^i) \cap K|K) \geq C_0 \xi(T);$$

(2c) for each component K of X , we have

$$\xi(\Lambda_g(V) \cap K|K) \geq C_0 \xi(T);$$

(2d) for the interval V_0 , we have

$$\xi((\Lambda_g(V) \cup V) \cap V_0|V_0) \geq C_0 \xi(T);$$

(2e) for each component K of Q' , we have

$$\xi(\Lambda_g(V) \cap K|K) \geq C_0 \xi(T).$$

Proof of Claim 2. (2a) follows from Lemma 13.

(2b) follows from (2a) and the observation that $K' = f^{is}(K)$ is a component of $E(T) \cap (T \setminus T^1)$ and $f^{is}(K \cap \Lambda_g(T^i)) \subset K' \cap \Lambda_g(T^i) \subset K' \cap \Lambda_g(T^1)$.

(2c) follows similarly.

(2d). The set $(\Lambda_g(V) \cup V) \cap V_0$ is covered by V and the components of $X \cap V_0$. The statement follows from (1b) and (2c) by Lemma 11.

(2e) follows from (2c) and (2d) by the observation on Q' . \square

Now suppose $J \subsetneq T^m$ is a child of $T^{m'}$ for some $0 \leq m' \leq m$. Let t be a transition time from J to $T^{m'}$. Note that $J \subset V$, so

$$Y := f^t(\Lambda_g(J) \cap J) \subset (\Lambda_g(J) \cup J) \cap T^{m'} \subset (\Lambda_g(V) \cup V) \cap T^{m'}.$$

Let U_0, U_1, U_2, \dots be the components of $E(T) \cap T^{m'}$, $X \setminus \overline{Q \cup V_0}$, V_0 and Q' . These sets cover Y . Each of these intervals are uniformly well inside $T^{m'}$ and $\xi(Y|U_i) \geq C_0 \xi(T)$. By Lemma 12, it follows that $\xi(J) \geq C \xi(T)$, where $C > 0$ is a constant.

We have completed the proof of the proposition under the assumption (*). For the general case, by Proposition 3, we may assume $m \geq m' \geq 2$, so $T^1 \supset T^2 \supset \dots \supset T^m$ is also a maximal central cascade. We claim that each component K of $D(T^1) \cap (T^1 \setminus T^2)$ is τ_1 -well inside $T^1 \setminus T^2$ for some $\tau_1 > 0$. Indeed, the first return time of K into T^1 is greater than s . Since T^1 is well inside T , $f^s(K)$ is well inside a component K' of $D(T)$, by Theorem 3.

Thus $f^s(K)$ is well inside $T \setminus T^1$, which implies that K is well inside $T^1 \setminus T^2$. Applying the above argument to the maximal central cascade $T^1 \supset T^2 \supset \dots \supset T^m$ proves the statement. \square

5. PROOF OF THE REDUCED MAIN THEOREM

We continue to assume that f is non-renormalizable and has a Cantor attractor $\omega(c)$. Fix a nice interval $I \ni c$ as in the previous section.

Let $Y \ni c$ be a symmetric nice interval which is contained in $(f(c), f^2(c))$. Let $\mathcal{N}_Y = \{n \geq 1 : f^n(c) \in Y\}$ and for each $n \in \mathcal{N}_Y$, let Y_{-n} denote the pull back of Y by f^n which contains c . Since f is non-renormalizable, we have

$$(18) \quad \lim_{\substack{n \in \mathcal{N}_Y \\ n \rightarrow \infty}} |Y_{-n}| = 0.$$

For $n \in \mathcal{N}_Y$ with $n \geq 1$, we shall define a positive integer $M_n(Y)$, called the *essential order* of Y_{-n} . Let $\{Y_i\}_{i=-n}^0$ be the chain with $Y_0 = Y$. Let $0 = i_0 > i_1 > \dots > i_p = -n$ be all the integers such that $Y_{i_j} \ni c$. So Y_{i_j} is a child of $Y_{i_{j-1}}$ with the transition time $s_j = i_{j-1} - i_j$, for each $1 \leq j \leq p$. By Lemma 1, $s_1 \leq s_2 \leq \dots \leq s_p$. Define

$$M_n(Y) = \#\{s_j : 1 \leq j \leq p\}.$$

Let

$$\mathcal{N}_M(Y) = \{n \in \mathcal{N}_Y : M_n(Y) \leq M\}.$$

Proposition 5. *There exists a universal constant $C_0 > 0$ such that for any symmetric nice interval $Y \ni c$ and $n \in \mathcal{N}_Y$, we have $M_n(Y) \leq C_0 \log n$.*

Proof. Let $\{Y_i\}_{i=-n}^0$, i_j and s_j be defined as above, and let $M = M_n(Y)$. Define $m(0) = 0$, $m(1) = 1$, and define inductively integers $m(1) < m(2) < \dots < m(M) \leq p$ by

$$m(j) = \inf\{m > m(j-1) : s_m > s_{m(j-1)}\}, \quad j = 2, 3, \dots$$

For $1 \leq j \leq M$, let r_j denote the minimal return time of points in $Y_{i_{m(j)-1}} \cap \omega(c)$ to $Y_{i_{m(j)-1}}$. Then by Lemma 1, for $2 \leq j \leq M$, $r_j \geq s_{m(j-1)}$. Moreover, since $s_{m(j)} - s_{m(j-1)}$ is a return time of $f^{s_{m(j-1)}}(c)$ into $J_{m(j-1)-1}$, we have

$$s_{m(j)} \geq s_{m(j-1)} + r_{j-1}.$$

Writing $s_0 = r_0$, we obtain that

$$s_{m(j)} \geq s_{m(j-1)} + s_{m(j-2)}.$$

Thus $s_{m(j)}$ grows at least as fast as the Fibonacci sequence. Since $n \geq s_{m(M)}$, it follows that $M \leq C_0 \log n$ for some universal constant $C_0 > 0$. \square

Lemma 14. *Given a symmetric nice interval Y , for each $M \geq 1$, we have $\#\mathcal{N}_M(Y) < \infty$.*

Proof. It suffices to prove that $\mathcal{N}_1(Y)$ is finite, since for each $n \in \mathcal{N}_M(Y)$ with $M \geq 2$, there exists $n' \in \mathcal{N}_{M-1}(Y)$ such that $n \in \mathcal{N}_1(Y_{-n'})$.

By Proposition 1, the set

$$\mathcal{N}_1^o(Y) = \{s : Y_{-s} \text{ is a child of } Y\}$$

is finite. For each $n \in \mathcal{N}_1(Y) \setminus \mathcal{N}_1^o(Y)$, there exists $s \in \mathcal{N}_1^o(Y)$ such that $n - s \in \mathcal{N}_1(Y)$ and such that Y_{-n} is a child of Y_{-n+s} . Since $|Y_{-n+s}| \geq |f^s(c) - c|$ is bounded away from zero, by (18), $n - s$ is bounded from above. It follows that $\mathcal{N}_1(Y) \setminus \mathcal{N}_1^o(Y)$, hence $\mathcal{N}_1(Y)$, is finite. \square

In particular, for each symmetric nice interval $Y \subset I$,

$$\widehat{\xi}_M(Y) = \inf\{\xi(Y_{-n}) : n \in \mathcal{N}_M(Y)\} > 0.$$

Lemma 15. *Let $Y \ni c$ be a small symmetric nice interval, let $n \in \mathcal{N}_Y$ be such that $M_n(Y) \geq 3$, and let Y_{-n} be the critical pull back of Y by f^n . Let $K_1 \supset K_2 \supset \dots$ be all the children of Y_{-n} . Then for each $k \geq 2$, K_k is $C\lambda_0^{-k}$ -well inside Y_{-n} and*

$$(19) \quad \xi(K_k) \geq n^{-\beta} \widehat{\xi}_2(Y),$$

where $C > 0$, $\lambda_0 \in (0, 1)$ and $\beta > 0$ are universal constants.

Proof. Let $\{Y_i\}_{i=-n}^0$ be the chain with $Y_0 = Y$ and define $m(0), m(1), \dots$ as above. Define $T_j = Y_{i_j}$ for $0 \leq j \leq p$. Note that $T_{m(2)}$ is of the form $Y_{-n'}$ for some $n' \in \mathcal{N}_2(Y)$, so

$$\xi(T_{m(2)}) \geq \widehat{\xi}_2(Y).$$

Let us first prove that there exists a universal constant $\tau > 0$ such that for each $2 \leq j \leq M$, $T_{m(j)}$ is a τ -nice interval. If either $T_{m(j)}$ is well inside $T_{m(j)-1}$ or $T_{m(j)-1}$ is well inside $T_{m(j)-2}$ then by Lemma 2 we are done. Thus, by Lemma 3, we may assume that $T_{m(j)}$ is the first child of $T_{m(j)-1}$ and that $T_{m(j)-1}$ is the first child of $T_{m(j)-2}$, i.e. $s_{m(j)}$ is the first return time of c into $T_{m(j)-1}$ and $s_{m(j)-1}$ is the first return time of c into $T_{m(j)-2}$. Since $s_{m(j)} > s_{m(j)-1}$, it follows that $R_{T_{m(j)-2}} : T_{m(j)-1} \rightarrow T_{m(j)-2}$ is non-central. By Theorem 2, it follows that $T_{m(j)}$ is uniformly well inside $T_{m(j)-1}$ and thus we are done.

Now let us show that there exists $\kappa \in (0, 1)$ such that $\xi(T_{m(j)}) \geq \kappa \xi(T_{m(j-1)})$ for each $3 \leq j \leq M$. Indeed, by Proposition 3, such an estimate holds if $m(j) = m(j-1) + 1$. So assume $m(j) > m(j-1) + 1$. By Lemma 1, it follows that $T_{m(j-1)} \supset T_{m(j-1)+1} \supset \dots \supset T_{m(j)-1}$ is a central cascade, i.e., $s_k = s_{m(j-1)}$ is the first return time to T_{k-1} for each $m(j-1) < k \leq m(j)-1$. Since $s_{m(j)} > s_{m(j-1)}$, $T_{m(j)}$ is an inheritor of $T_{m(j-1)}$. So by Proposition 4, the statement follows.

Similarly for each $k \geq 2$, K_k is either a child or an inheritor of $T_{m(M)}$, so $\xi(K_k) \geq \kappa \xi(T_{m(M)})$. Thus

$$\xi(K_k) \geq \kappa^{M-1} \xi(T_{m(2)}) \geq \kappa^{M-1} \widehat{\xi}_2(Y).$$

By Proposition 5, the statement follows.

If $Y_{-n} = T_{m(M)}$, then by Lemma 6, the children of Y_{-n} are well nested, so the niceness of K_k follows from Lemma 2. If $Y_{-n} \subsetneq T_{m(M)}$, then for each $k \geq 2$, the same conclusion follows from Proposition 2. \square

Proof of the Reduced Main Theorem. By (18) and Lemma 14, we may assume that Y is small so that Lemma 15 applies.

Let $n \in \mathcal{N}_Y$ be so large that $M_n(Y) \geq 3$ and Y_0 be the critical pull back of Y under f^n . Assume the number N of children of Y_0 is at least 2 and let K_N denote the N -th child of Y_0 . Let L be the first child of K_N . Then by Lemma 15, $\xi(K_N) \geq n^{-\beta} \widehat{\xi}_2(Y)$ and K_N is a $C\lambda_0^{-N}$ -nice interval, where $C > 0$ and $\lambda_0 \in (0, 1)$ are universal constants. So by Proposition 3, we have

$$\xi(L) \geq \frac{1 - C_0\lambda_0^{N/\ell}}{\xi(K_N) + C_0\lambda_0^{N/\ell}} \xi(K_N) \geq \frac{1 - C_0\lambda_0^{N/\ell}}{\widehat{\xi}_2(Y) + C_0n^\beta\lambda_0^{N/\ell}} \widehat{\xi}_2(Y),$$

where $C_0 > 0$ is a constant. On the other hand, by Lemma 10, when n is large enough, we have $\xi(L) \leq \widehat{\xi}_2(Y)/2$. Since $\widehat{\xi}_2(Y) \leq 1$, it follows that $N = O(\log n)$. \square

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